Strategic Voting

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Situations where privately informed agents collectively make a decision without using transfers:

- Electing a candidate in elections.
- Various departments of a university jointly choosing a hiring policy.
- Organizing committee of a conference choosing a speaker from a list of speakers.
- Setting the temperature of a classroom.

Implications of No Transfers

Designer cannot transfer utility (by payments) from agents.

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Strategy-proofness will mean much of the aggregated private information of agents is not used in many environment. Two extreme illustrations:

- dictatorship
- median of tops

Why Begin Here?

Assumptions make sense in a variety of environments.

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The Benchmark Model.

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Implications of domain restrictions (smaller type space) easily spelt out.

The Unrestricted Domain Model

- A set of agents $N = \{1, \ldots, n\}$.
- A set of alternatives $A = \{a, b, \ldots\}$ - assume A to be finite.
- Type of agent i: strict ordering P_i of A.
- Domain or type space of each agent: set of all strict orderings of A, denoted by $\mathcal{P}$.

Social Choice Functions (scfs)

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Note no randomization.

Two Alternative Example

P_1	P_2	P_3
a	b	a
b	a	b

Table: Majority scf with two alternatives.

Condorcet Paradox - Three Alternatives

P_1	P_2	P_3
a	b	c
b	c	a
c	a	b

Table: Condorcet

Plurality Voting

P_1	P_2	P_3	P'_1	P'_2	P'_3
a	b	a	a	b	c
b	c	c	b	c	a
c	a	b	c	a	b

Table: Plurality scf.

Borda Voting

P_1	P_2	P_3	P'_1	P'_2	P'_3
a	b	b	c	b	b
c	c	c	a	c	c
b	a	a	b	a	a

Table: Borda scf.

Strategy-proofness

Reporting true type is a weakly dominant strategy.

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Definition

An scf f is **strategy-proof** *if for every $i \in N$, for every $P_{-i} \in \mathcal{P}_{-i}$, for every $P_i \in \mathcal{P}$, there exists no $P'_i \in \mathcal{P}$ such that*

$$f(P'_i, P_{-i}) \;\; P_i \;\; f(P_i, P_{-i}).$$

Example - Two Alternatives

P_1	P_2	P_3
a	b	a
b	a	b

Table: Majority scf with two alternatives.

Plurality scf

P_1	P_2	P_3	$P'_1 = P_1$	$P'_2 = P_2$	P'_3
a	b	c	a	b	b
b	c	a	b	c	a
c	a	b	c	a	c

Table: Plurality scf is manipulable.

Plurality scf

P_1	P_2	P_3	$P'_1 = P_1$	$P'_2 = P_2$	P'_3
a	b	c	a	b	b
b	c	a	b	c	a
c	a	b	c	a	c

Table: Plurality scf is manipulable.

Plurality scf belongs to a broad class of scfs called scoring rules that are all manipulable.

Strategy-proof scfs

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Notation: $P_i(k)$: k-th ranked alternative according to P_i.

Definition

An scf f is a **dictatorship** *scf if there exists an agent $i \in N$ such that at every profile $P \in \mathcal{P}$, we have $f(P) = P_i(1)$.*

Monotonicity

For any alternative $a \in A$, let $B(a, P_i)$ be the set of alternatives below a in preference ordering P_i. Formally, $B(a, P_i) := \{b \in A : aP_ib\}$.

Definition

A social choice function f is **monotone** *if for any two profiles P and P' with $B(f(P), P_i) \subseteq B(f(P), P'_i)$ for all $i \in N$, we have $f(P) = f(P')$.*

Illustrating Monotonicity

P_1	P_2	P_3	P'_1	P'_2	P'_3
x	b	x	x	x	c
a	x	c	a	a	a
b	a	a	c	c	x
c	c	b	b	b	b

Table: Two valid profiles for monotonicity

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b	a	a	c	c	x
c	c	b	b	b	b

Table: Two valid profiles for monotonicity

Note: no restriction is imposed on scf at profiles other than such monotonic transformation profiles.

Equivalence to Monotonicity

Theorem

Every strategy-proof scf satisfies monotonicity. Conversely, in the unrestricted domain, every monotone scf is strategy-proof.

Strategy-proof implies monotonicity

Start from $P \equiv (P_1, P_2, \ldots, P_n)$ with $f(P) = a$ and $P' \equiv (P'_1, P'_2, \ldots, P'_n)$. Assume that $B(a, P_i) \subseteq B(a, P'_i)$ for all $i \in N$.

Consider $P'' \equiv (P'_1, P_2, \ldots, P_n)$. Suppose $f(P'') = b \neq a$.

If aP_1b then aP'_1b. In that case, agent 1 manipulates from P'_1 to P_1.

If bP_1a then agent 1 manipulates from P_1 to P'_1.

Monotonicity implies strategy-proofness

Suppose agent i can manipulate at preference profile P by a preference ordering P_i'.

Suppose $f(P_i, P_{-i}) = a$ and $f(P_i', P_{-i}) = b$, and by assumption bP_ia.

Consider a preference profile $P'' \equiv (P_i'', P_{-i})$, where P_i'' is any preference ordering satisfying $P_i''(1) = b$ and $P_i''(2) = a$.

By monotonicity, $f(P'') = f(P') = b$ and $f(P'') = f(P) = a$

Other Normative/Technical Properties

Definition

An scf f is **unanimous** *if for every $P \in \mathcal{P}^n$ with $P_1(1) = P_2(1) = \ldots = P_n(1)$, we have $f(P) = P_1(1)$.*

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An scf f is **onto** *if for every $a \in A$, there exists a $P \in \mathcal{P}^n$ such that $f(P) = a$.*

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Definition

An scf f is **onto** *if for every $a \in A$, there exists a $P \in \mathcal{P}^n$ such that $f(P) = a$.*

Definition

An scf f is **Pareto efficient** *if for every $P \in \mathcal{P}^n$ and for every $a \in A$, if there exists $b \in A$ such that bP_ia for all $i \in N$, then $f(P) \neq a$.*

Gibbard-Satterthwaite Theorem

Theorem (Gibbard 1973, Satterthwaite 1975)

Suppose $|A| \geq 3$ *and* $f : \mathcal{P}^n \rightarrow A$ *is an scf. Then,* f *satisfies unanimity and strategy-proofness if and only if it is a dictatorship.*

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- Does not hold if $|A| = 2$.
- Possible to state: suppose range of f is at least three.
- May not hold if domain is smaller than $\mathcal{P}$ - more later.
- Allowing for indifferences is fine as long as we allow for strict orderings - though dictatorship may no longer be strategy-proof.

Dictatorship with Indifferences

P_1	P_2	P'_1	P'_2
x, a	b	x, a	x
b	x	b	b
c	a	c	a
	c		c

Table: Two valid profiles for monotonicity

Proof Technique

Many proofs ...

- Earlier (original) proofs were based on using Arrow's impossibility theorem.
- Later many independent proofs - Barbera, Reny, Benoit, Svensson.
- Proof for two agents case - then do induction on number of agents. (Proof due to Sen (2001))

Two Agent Proof Idea

Lemma (Top Selection Lemma)

Suppose $|A| \geq 3$ *and* $N = \{1, 2\}$. *Suppose* f *is unanimous and strategy-proof social choice function. Then for every preference profile* P, $f(P) \in \{P_1(1), P_2(1)\}$.

Two Agent Proof Idea

Lemma (Top Selection Lemma)

Suppose $|A| \geq 3$ and $N = \{1, 2\}$. Suppose f is unanimous and strategy-proof social choice function. Then for every preference profile P, $f(P) \in \{P_1(1), P_2(1)\}$.

P_1	P_2	P_1	P_2'	P_1'	P_2'	P_1'	P_2
a	b	a	b	a	b	a	b
$\cdot$	$\cdot$	$\cdot$	a	b	a	b	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Table: Preference profiles.

Dictatorship Lemma

Lemma

Suppose $|A| \geq 3$ *and* $N = \{1, 2\}$. *Suppose* f *is unanimous and strategy-proof social choice function. Consider a profile* P *such that* $P_1(1) = a \neq b = P_2(1)$. *If* $f(P) = P_i(1)$ *for some* $i \in N$, *then* $f(P') = P_i(1)$ *for all* P'.

Case 1 - $c = a$, $d = b$

Tops-only property.

P_1	P_2	P'_1	P'_2	$\hat{P}_1$	$\hat{P}_2$
a	b	a	b	a	b
.	.	.	.	b	a
.	.	.	.	.	.

Table: Preference profiles required in Case 1.

Case 1 - $c = a$, $d = b$

Tops-only property.

P_1	P_2	P'_1	P'_2	$\hat{P}_1$	$\hat{P}_2$
a	b	a	b	a	b
.	.	.	.	b	a
.	.	.	.	.	.

Table: Preference profiles required in Case 1.

Monotonicity from two different profiles.

Case 2 - $c \neq a$, $d = b$

P_1	P_2	P'_1	P'_2	$\hat{P}_1$	P_2
a	b	$c \neq a$	$d = b$	c	b
.	.	.	.	a	.
.	.	.	.	.	.

Table: Preference profiles required in Case 2.

Case 2 - $c \neq a$, $d = b$

P_1	P_2	P'_1	P'_2	$\hat{P}_1$	P_2
a	b	$c \neq a$	$d = b$	c	b
.	.	.	.	a	.
.	.	.	.	.	.

Table: Preference profiles required in Case 2.

Case 1 from 2nd to third profile and strategy-proofness from first to third profile.

Case 2 - $c \neq a$, $d = b$

P_1	P_2	P'_1	P'_2	$\hat{P}_1$	P_2
a	b	$c \neq a$	$d = b$	c	b
.	.	.	.	a	.
.	.	.	.	.	.

Table: Preference profiles required in Case 2.

Case 1 from 2nd to third profile and strategy-proofness from first to third profile.

Other cases similarly resolved except for one.

Case - $c = b$, $d = a$

P_1	P_2	P'_1	P'_2	$\hat{P}_1$	P'_2	$\hat{P}'_1$	P'_2
a	b	$c = b$	$d = a$	b	a	x	a
.	.	.	.	x	.	.	.
.	.	.	.	.	.	.	.

Table: Preference profiles required in Case 6.

Case - $c = b$, $d = a$

P_1	P_2	P'_1	P'_2	$\hat{P}_1$	P'_2	$\hat{P}'_1$	P'_2
a	b	$c = b$	$d = a$	b	a	x	a
.	.	.	.	x	.	.	.
.	.	.	.	.	.	.	.

Table: Preference profiles required in Case 6.

First to fourth profile is handled by earlier cases. Other profiles by Case 1 and strategy-proofness.

Idea for Induction

- From a n-agent scf, construct a $(n-1)$-agent scf by considering outcome of the scf where two agents have the same preference.
- Show that the $(n-1)$ agent scf is unanimous and strategy-proof, and conclude dictatorship of it.
- Use this to argue that the original scf is also a dictatorship - this step will require that $n \geq 3$ (hence, induction must start at $n = 2$).

How to Escape Impossibility?

Domain Restriction. Two common ways to do it:

- Type space is restricted.
- Randomization is considered - outcomes are lotteries. Ranking of lotteries usually done in a specific way - again leading to domain restriction.

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Weaken solution concept.

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Weaken solution concept.
Weaken rationality - agents may not manipulate to all possible types.

Single Peaked Domain

In many settings, not all possible strict orderings may be a type.

Alternatives are often ordered - days of a week, locations along a street, readings of temperature, political ideology of candidates.

Agents have ideal point on the ordered set and their preference for alternatives become worse as they go away from the ideal - quasiconcave.

A firm wants delivery on Wednesday as ideal and as one goes away from Wednesday, his preferences become worse - so he never likes Friday to Thursday.

Number of Single Peaked Orderings

$a \succ b \succ c \succ d.$

a	b	b	b	c	c	c	d
b	a	c	c	d	b	b	c
c	c	d	a	b	a	d	b
d	d	a	d	a	d	a	a

Table: Single-peaked preferences

Number of Single Peaked Orderings

$a \succ b \succ c \succ d$.

a	b	b	b	c	c	c	d
b	a	c	c	d	b	b	c
c	c	d	a	b	a	d	b
d	d	a	d	a	d	a	a

Table: Single-peaked preferences

Note: No restriction on alternatives on either side of the peak.

A Formal Definition

A preference ordering P_i of agent i is **single peaked** with respect to $\succ$ if for all $b, c \in A$,

- with $b \succ c \succ P_i(1)$ we have cP_ib, and
- with $P_i(1) \succ b \succ c$ we have bP_ic.

Possibility in Single Peaked Domain

Consider the following SCF f: for every preference profile P, $f(P)$ is the minimal element with respect to $\succ$ among $\{P_1(1), P_2(1), \ldots, P_n(1)\}$.

Why is this strategy-proof?

Possibility in Single Peaked Domain

Consider the following SCF f: for every preference profile P, $f(P)$ is the minimal element with respect to $\succ$ among $\{P_1(1), P_2(1), \ldots, P_n(1)\}$.

Why is this strategy-proof?

- Agent whose peak coincides with the chosen alternative has no incentive to deviate.
- If some other agent deviates, then the only way to change the outcome is to place his peak to the left of chosen outcome.
- But that will lead to an outcome which is even more left to his peak, which he prefers less than the current outcome.

A Class of Possibilities

Pick an integer $k \in \{1, \ldots, n\}$. In every preference profile, the SCF picks the k-th lowest peak according to $\succ$.

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Pick an integer $k \in \{1, \ldots, n\}$. In every preference profile, the SCF picks the k-th lowest peak according to $\succ$.

- Note that those agents whose peak coincides with the k-th lowest peak have no incentive to manipulate.
- Consider an agent i, which lies to the left of k-th lowest peak. The only way he can change the outcome is to move to the right of the k-th lowest peak.
- In that case, an outcome which is even farther away from his peak will be chosen. According to single-peaked preferences, he prefers this less.
- A symmetric argument applies to the agents who are on to the right of k-th lowest peak.

Median Voter SCF

Definition

A social choice function $f : \mathcal{S}^n \to A$ *is a* **median voter** *social choice function if there exists* $B = (y_1, \ldots, y_{n-1})$ *such that* $f(P) = median(B, P_1(1), P_2(1), \ldots, P_n(1))$ *for all preference profiles* P*. The alternatives in* B *are called the peaks of* **phantom voters***.*

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Locating $(n-1)$ phantom peaks at different alternatives give different scfs.

Examples of Median Voter SCF

Suppose $a \succ b \succ c$ and with three agents. Put both the phantoms at a. Then, the scf chooses the minimum of the peaks.

Suppose one phantom is at a and the other is at c. Then, it chooses the median of the voter peaks.

Suppose both the phantoms are at b, then unless there is unanimity, we choose b.

Strategy-proof Median Voter

Theorem

Every median voter social choice function is strategy-proof.

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Every median voter social choice function is strategy-proof.

- Agent i has no incentive to manipulate if $P_i(1) = f(P) = a$.
- Suppose agent i's peak is to the left of a.
- The only way he can change the outcome is by changing the median, which he can only do by changing his peak to the right of a.
- But that will shift the median to the right of a which he does not prefer to a. So, he cannot manipulate.
- A symmetric argument applies if i's peak is to the right of a.

Anonymous scf

Identity of agents do not matter - dictatorship is not anonymous.

Definition

A social choice function $f : \mathcal{S}^n \to A$ *is* **anonymous** *if for every profile* P *and every permutation* σ *such that* $P^\sigma \in \mathcal{S}^n$*, we have* $f(P^\sigma) = f(P)$*.*

Illustration

P_1	P_2	P_3	P'_1	P'_2	P'_3
x	b	b	b	x	b
a	a	a	a	a	a
b	x	c	c	b	x
c	c	x	x	c	c

Table: Anonymity

Characterization Result

Theorem

A strategy-proof social choice function is unanimous and anonymous if and only if it is the median voter social choice function.

Condorcet Winner Exists and Strategy-proof

- With odd number of agents, an alternative exists that beats every other alternative in a pair-wise majority.
- An scf choosing such a Condorcet winner is strategy-proof.
- It is a median voter scf where phantoms (even in number) are equally distributed between the two extreme alternatives and Condorcet winner is the median of the agent peaks.

Concluding Thoughts

- Strategy-proofness is too strong in the unrestricted domain.
- Long literature to characterize possibility in restricted domains - possibility domains are variants of single peaked domain.
- Private good allocation brings domain restrictions and indifferences - e.g., matching.
- Randomization also brings domain restrictions - more possibilities.
- Considering **local** strategy-proofness instead of strategy-proofness brings nothing new.